Int. J. Solids Structures Vol. 22, No. 7, pp. 737–758, 1986
Printed in Great Britain.
0020-7683/86 $3.00+.00
Pergamon Journals Ltd.

ON A CLASS OF CONSERVATION RULES ASSOCIATED WITH SENSITIVITY ANALYSIS IN LINEAR ELASTICITY

K. DEMS
Łódź Technical University, Łódź, Poland

and

Z. MRÓZ
Institute of Fundamental Technological Research, Warsaw, Poland

(*Received* 25 *September* 1984)

Abstract—Considering arbitrary stress, strain or displacement functionals specified over a domain of an elastic, homogeneous and isotropic body, their invariance is proved for the case of translation, rotation and scale change of an arbitrary domain within the body. The assosciated class of path-independent integrals is derived. It is shown that sensitivity analysis with respect to translation, rotation or expansion of defects can be performed by using these path-independent integrals.

1. INTRODUCTION

The present paper discusses a new class of conservation rules which constitute an extension of the class considered by Eshelby[1, 2], Günther[3], Knowles and Sternberg[4], Gołębiewska-Herrmann[7], Delph[9], Rice[12] and Bui[14] for linear and nonlinear elasticity. Whereas the previous rules were associated with the potential or complementary energy variation due to translation, rotation or scale change of the body, the present analysis is concerned with an arbitrary functional of stress, strain or displacement. Similar to the Eshelby[1, 2] or Budiansky and Rice[13] interpretation, the variation of respective functionals can be interpreted as that corresponding to translation, rotation or size variation of inhomogeneities within the body. Therefore the derived conservation rules can find application in identification problems where the location and size of defects are to be determined for some mechanical measurements, or in studying the variation of global or local body response due to variation of position and size of defects such as cracks, voids and inclusions. The general formulation of such a sensitivity analysis problem was presented in previous papers by Dems and Mróz[15, 16], and here this analysis will be extended by discussing three types of path-independent integrals and their interpretations.

In discussing the conservation rules, Knowles and Sternberg[4] demonstrated that such laws follow from Noether's theorem[6] on invariant variational principles combined with the principle of stationary potential energy. In our case, the functionals considered do not possess this stationarity property and therefore the derived conservation rules are not directly generated from Noether's theorem,† thus constituting a new class of rules. The concept of primary and adjoint systems will be used and the conservation rules will be expressed in terms of stress and strain fields of both systems. A somewhat similar idea of introducing adjoint variables was discussed recently by Herrmann[11] and Delph[10] who considered a nonlinear creep problem for which an energy stationarity principle does not exist. In particular, it will be shown that bilinear functionals of primary and adjoint variables can also be considered within the considered class of functionals. Since the local displacement or stress components can be expressed as bilinear functionals, their variation can be derived through the use of path-independent integrals. The conservation rules for the mutual potential energy of two equilibrium states were discussed by Chen and Shield[18] and applied in fracture mechanics for the determination of stress intensity factors K^{I}, K^{II}

† They can be derived from Noether's theorem by considering augmented functionals which take into account the equilibrium and compatibility equations of the body.

in two modes. The same problem was reanalysed by Bui[19], who proposed an alternative method of solution using the associated path-independent integrals.

The conservation rules in elastodynamics were discussed by Fletcher[5], whereas Francfort and Gołębiewska-Herrmann[8] derived the conservation laws in thermoelasticity using the convolution products of primary and adjoint states. It can be shown that the present approach can easily be extended to elastodynamics and to time-dependent problems, but in the present paper we limit our analysis to the case of elastostatics.

In Section 2, the general expression for variation of arbitrary volume or surface integrals of stress, strain and displacements due to boundary variation will be derived, and the concept of an adjoint body will be introduced following the previous works by Dems and Mróz[15, 16]. In Section 3, the respective conservation rules will be proved for the case of linear elasticity and small strain theory. In Section 4, the transition to conservation laws discussed in Refs. [1–4, 14, 18] will be performed, whereas in Section 6 some possible applications will be indicated.

2. VARIATION OF VOLUME AND SURFACE INTEGRALS DUE TO BOUNDARY VARIATION

Consider a linear elastic body with surface tractions $\mathbf{T}^0$ specified on its boundary portion S_T and displacements $\mathbf{u}^0$ prescribed on the portion S_u, where $S = S_T \cup S_u$ denotes the boundary of the body. We shall discuss the variation of functionals

$$G_1 = \int \psi(\boldsymbol{\sigma}, \mathbf{u})\, \mathrm{d}V + \int h(\mathbf{T}, \mathbf{u})\, \mathrm{d}S \tag{1}$$

or

$$G_2 = \int \phi(\boldsymbol{\varepsilon}, \mathbf{u})\, \mathrm{d}V + \int h(\mathbf{T}, \mathbf{u})\, \mathrm{d}S \tag{2}$$

specified over the body domain of volume V, associated with the variation of its boundaries. Here $\psi(\boldsymbol{\sigma}, \mathbf{u})$, $\phi(\boldsymbol{\varepsilon}, \mathbf{u})$ and $h(\mathbf{T}, \mathbf{u})$ are continuous and differentiable functions of their arguments. The stress, strain and displacement fields are denoted by $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ and $\mathbf{u}$, where $\mathbf{u}(\mathbf{x})$ is a continuous field and $\boldsymbol{\sigma}(\mathbf{x})$, $\boldsymbol{\varepsilon}(\mathbf{x})$ are piecewise continuous fields. The particular case when $\psi(\boldsymbol{\sigma}, \mathbf{u}) = \psi_1(\boldsymbol{\sigma}) + \psi_2(\mathbf{u})$, $h(\mathbf{T}, \mathbf{u}) = h_1(\mathbf{T}) + h_2(\mathbf{u})$, $\phi(\boldsymbol{\varepsilon}, \mathbf{u}) = \phi_1(\boldsymbol{\varepsilon}) + \phi_2(\mathbf{u})$ was discussed in previous papers[15,16]. We assume this particular case in Section 3 when considering rotation and scale change of body domain.

The variation of body shape is conceived as the transformation process specified by the transformation field $\boldsymbol{\varphi}(\mathbf{x})$ mapping the material points from an initial to a transformed configuration, $P \to P_t$: $\mathbf{x}_t = \mathbf{x} + \boldsymbol{\varphi}$. In this paper, we shall restrict our analysis to an infinitesimal transformation $\delta\boldsymbol{\varphi}(\mathbf{x})$ from the assumed configuration and derive the formulae for variation of the functionals G_1 and G_2 associated with this transformation. If $\mathbf{x}^*$ denotes the position of a point P, initially placed at $\mathbf{x}$, after infinitesimal variation $\delta\boldsymbol{\varphi}(\mathbf{x})$, we have

$$P \to P^*: \; x_i^* = x_i + \delta\varphi_i \tag{3}$$

and the variations of displacement, stress and strain fields are expressed in a fixed reference system as follows:

$$\begin{gathered} \delta u_i = u_i^*(\mathbf{x}^*) - u_i(\mathbf{x}) = \delta\bar{u}_i + u_{i,k}\delta\varphi_k, \\ \delta\varepsilon_{ij} = \delta\bar{\varepsilon}_{ij} + \varepsilon_{ij,k}\delta\varphi_k, \qquad \delta\sigma_{ij} = \delta\bar{\sigma}_{ij} + \sigma_{ij,k}\delta\varphi_k, \end{gathered} \tag{4}$$

where $\delta\bar{\mathbf{u}}$, $\delta\bar{\boldsymbol{\varepsilon}}$ and $\delta\bar{\boldsymbol{\sigma}}$ denote the variations at the initial configuration of the body, and subscripts following commas denote partial derivatives with respect to coordinates of the Cartesian system. Clearly, the variations $\delta\bar{\mathbf{u}}$, $\delta\bar{\boldsymbol{\varepsilon}}$, $\delta\bar{\boldsymbol{\sigma}}$ can be determined by considering an incremental boundary value problem which accounts for the variation of boundary conditions due to the boundary surface transformation. The variations of surface tractions and of volume and surface elements are, cf. Ref. [16],

$$\begin{aligned} \delta T_i(\mathbf{x}) &= \delta \bar{T}_i(\mathbf{x}) + (\sigma_{ij}n_jn_l - \sigma_{il})n_k\delta\varphi_{k,l} + \sigma_{ij,k}n_j\delta\varphi_k, \\ \delta[\mathrm{d}V(\mathbf{x})] &= \delta\varphi_{k,k}\,\mathrm{d}V, \\ \delta[\mathrm{d}S(\mathbf{x})] &= (\delta_{kl} - n_kn_l)\delta\varphi_{k,l}, \end{aligned} \tag{5}$$

where $\mathbf{n}$ denotes the unit normal vector to the boundary surface.

In view of (5), the variation of the functional G_1 corresponding to an infinitesimal transformation of the body domain is expressed as follows:

$$\begin{aligned} \delta G_1 = \int\left(\frac{\partial\psi}{\partial\boldsymbol{\sigma}}\cdot\delta\bar{\boldsymbol{\sigma}} + \frac{\partial\psi}{\partial\mathbf{u}}\cdot\delta\bar{\mathbf{u}}\right)\mathrm{d}V + \int\psi n_k\delta\varphi_k\,\mathrm{d}S + \int\bigg\{\frac{\partial h}{\partial\mathbf{u}}\cdot\delta\bar{\mathbf{u}} + \frac{\partial h}{\partial\mathbf{u}}\cdot\mathbf{u}_{,k}\delta\varphi_k \\ + \frac{\partial h}{\partial\mathbf{T}}\cdot\delta\bar{\mathbf{T}} + \frac{\partial h}{\partial T_i}\left[(\sigma_{ij}n_jn_k - \sigma_{il})n_k\delta\varphi_{k,l} + \sigma_{ij,k}n_j\delta\varphi_k\right] \\ + h(\delta_{kl} - n_kn_l)\delta\varphi_{k,l}\bigg\}\,\mathrm{d}S. \end{aligned} \tag{6}$$

The expression for δG_1 contains the variations $\delta\bar{\mathbf{u}}(\mathbf{x})$, $\delta\bar{\boldsymbol{\sigma}}(\mathbf{x})$ and $\delta\bar{\mathbf{T}}(\mathbf{x})$, which should be determined from an additional boundary-value problem within the unperturbed domain which accounts for modification of the boundary conditions on S. This modification depends on a form of boundary variation. In fact, in view of (5), on S_T and S_u we have

$$\delta T_i^0 = \delta\bar{T}_i^0 + (\sigma_{ij}n_jn_l - \sigma_{il})n_k\delta\varphi_{k,l} + \sigma_{ij,k}n_j\delta\varphi_k, \qquad \delta u_i^0 = \delta\bar{u}_i^0 + u_{i,k}\delta\varphi_k. \tag{7}$$

Assuming, for instance, configuration-independent loading and support conditions, we have $\delta T_i^0 = \delta u_i^0 = 0$ and eqns (7) provide the values of $\delta\bar{T}_i^0$ and $\delta\bar{u}_i^0$ on S_T and S_u when boundary modification occurs, thus

$$\delta\bar{T}_i^0 = -(\sigma_{ij}n_jn_l - \sigma_{il})n_k\delta\varphi_{k,l} + \sigma_{ij,k}n_j\delta\varphi_k, \qquad \delta\bar{u}_i^0 = -u_{i,k}\delta\varphi_k, \tag{8}$$

and (8) provide new boundary conditions for a problem of determining $\delta\bar{\mathbf{u}}(\mathbf{x})$, $\delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$, $\delta\bar{\boldsymbol{\sigma}}(\mathbf{x})$ within the body.

Such a direct approach may become impractical in cases where the form of boundary modification changes and numerous solutions are required in order to determine variations $\delta\bar{\boldsymbol{\sigma}}$ and $\delta\bar{\mathbf{u}}$. An alternative approach used in sensitivity analysis requires introduction of an adjoint body and one solution of a boundary-value problem for this body. Following Refs. [15, 16], consider the adjoint elastic body of the same shape and material stress–strain relations, but satisfying the boundary conditions

$$\mathbf{T}^{a_0} = \frac{\partial h}{\partial\mathbf{u}} \quad \text{on} \quad S_T, \qquad \mathbf{u}^{a_0} = -\frac{\partial h}{\partial\mathbf{T}} \quad \text{on} \quad S_u, \tag{9}$$

and with imposed body force and initial strain fields

$$\mathbf{f}^a = \frac{\partial\psi}{\partial\mathbf{u}}, \qquad \boldsymbol{\varepsilon}^i = \frac{\partial\psi}{\partial\boldsymbol{\sigma}} \quad \text{within} \quad V. \tag{10}$$

Denoting the stress within the adjoint body by $\boldsymbol{\sigma}^r$, its total strain field $\boldsymbol{\varepsilon}^a$ can be presented as a sum, cf. Fig. 1

$$\boldsymbol{\varepsilon}^a = \boldsymbol{\varepsilon}^i + \boldsymbol{\varepsilon}^r \tag{11}$$

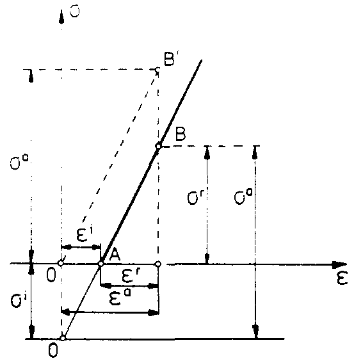

Fig. 1. Decomposition of strains and stresses in the adjoint body.

and is compatible with the displacement field $\mathbf{u}^a$. The stress field $\boldsymbol{\sigma}^r$ is related to $\boldsymbol{\varepsilon}^r$ by Hooke's law, $\boldsymbol{\sigma}^r = \mathbf{D}\cdot\boldsymbol{\varepsilon}^r = \mathbf{D}\,(\boldsymbol{\varepsilon}^a-\boldsymbol{\varepsilon}^i)$, and satisfies both equilibrium and boundary conditions

$$\operatorname{div}\boldsymbol{\sigma}^r+\mathbf{f}^a = 0 \quad \text{within} \quad V, \qquad \boldsymbol{\sigma}^r\cdot\mathbf{n} = \mathbf{T}^{a_0} \quad \text{on} \quad S_T, \tag{12}$$

whereas the displacement field $\mathbf{u}^a$ satisfies the boundary conditions $\mathbf{u}^a = \mathbf{u}^{a_0}$ on S_u. In view of (9)–(12), the first two terms of (6) can be transformed as follows:

$$\begin{aligned}\int\left(\frac{\partial\psi}{\partial\boldsymbol{\sigma}}\cdot\delta\bar{\boldsymbol{\sigma}}+\frac{\partial\psi}{\partial\mathbf{u}}\cdot\delta\bar{\mathbf{u}}\right)\mathrm{d}V &= \int(\boldsymbol{\varepsilon}^i\cdot\delta\bar{\boldsymbol{\sigma}}+\mathbf{f}^a\cdot\delta\bar{\mathbf{u}})\,\mathrm{d}V\\ &= \int(\boldsymbol{\varepsilon}^a\cdot\delta\bar{\boldsymbol{\sigma}}-\boldsymbol{\varepsilon}^r\cdot\delta\bar{\boldsymbol{\sigma}}+\mathbf{f}^a\cdot\delta\bar{\mathbf{u}})\,\mathrm{d}V\\ &= \int[\boldsymbol{\varepsilon}^a\cdot\delta\bar{\boldsymbol{\sigma}}-(\boldsymbol{\sigma}^r\cdot\delta\bar{\boldsymbol{\varepsilon}}-\mathbf{f}^a\cdot\delta\bar{\mathbf{u}})]\,\mathrm{d}V\\ &= \int\mathbf{u}^a\cdot\delta\bar{\mathbf{T}}\,\mathrm{d}S-\int\mathbf{T}^a\cdot\delta\bar{\mathbf{u}}\,\mathrm{d}S,\end{aligned} \tag{13}$$

where the following Betti's relations are applied: $\boldsymbol{\varepsilon}^r\cdot\delta\bar{\boldsymbol{\sigma}} = \boldsymbol{\varepsilon}^r\cdot\mathbf{D}\cdot\delta\bar{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma}^r\cdot\delta\bar{\boldsymbol{\varepsilon}}$, and $\mathbf{D}$ is the elastic stiffness matrix of both the primary and adjoint bodies. Using (13), the expression (6) for δG_1 can be presented in the form

$$\begin{aligned}\delta G_1 = \int\Bigg\{&\left[\psi n_k+\frac{\partial h}{\partial u_i}u_{i,k}+\frac{\partial h}{\partial T_i}\sigma_{ij,k}n_j\right]\delta\varphi_k+\Bigg[h(\delta_{kl}-n_kn_l)\\ &+\frac{\partial h}{\partial T_i}(\sigma_{ij}n_jn_l-\sigma_{il})n_k\Bigg]\delta\varphi_{k,l}\Bigg\}\mathrm{d}S+\int\left(\frac{\partial h}{\partial u_i}-\sigma^r_{ij}n_j\right)\delta\bar{u}_i^0\,\mathrm{d}S_u\\ &+\int\left(\frac{\partial h}{\partial T_i}+u_i^a\right)\delta\bar{T}_i^0\,\mathrm{d}S_T,\end{aligned} \tag{14}$$

where the local variations $\delta\bar{u}_i^0$ on S_u and $\delta\bar{T}_i^0$ on S_T are specified by (7). Let us note that δu_i^0 and δT_i^0 are known on the boundary portions S_u and S_T. The expression (14) for δG_1 now depends on stress and displacement fields of both the primary and adjoint bodies.

The variation of the functional G_2 specified by (2) is expressed similarly as in the previous case. Introducing the adjoint body subjected to the boundary conditions specified by (9) and with the initial stress and body force fields σ^i, $\mathbf{f}^a$ within V, such that

$$\sigma^i = \mathbf{D}\cdot\varepsilon^i = \frac{\partial\phi}{\partial\varepsilon}, \qquad \mathbf{f}^a = \frac{\partial\phi}{\partial\mathbf{u}}, \tag{15}$$

we obtain

$$\begin{aligned}\delta G_2 = \int\Big\{\Big[\phi n_k + \frac{\partial h}{\partial u_i}u_{i,k} + \frac{\partial h}{\partial T_i}\sigma_{ij,k}n_j\Big]\delta\varphi_k + \Big[h(\delta_{kl}-n_kn_l)\\ + \frac{\partial h}{\partial T_i}(\sigma_{ij}n_jn_l-\sigma_{il})n_k\Big]\delta\varphi_{k,l}\Big\}\,\mathrm{d}S + \int\Big(\frac{\partial h}{\partial u_i}-\sigma^r_{ij}n_j\Big)\delta\bar{u}^0_i\,\mathrm{d}S_u\\ + \int\Big(\frac{\partial h}{\partial T_i}+u^a_i\Big)\delta\bar{T}^0_i\,\mathrm{d}S_T.\end{aligned} \tag{16}$$

The expressions (14) and (16) now constitute the foundation for our subsequent analysis, in which the variations of G_1 and G_2 associated with translation, rotation and scale change of the body domain will be considered.

3. VARIATION OF G_1 AND G_2 ASSOCIATED WITH TRANSLATION, ROTATION AND SCALE CHANGE OF BODY DOMAIN

3.1. *Translation of body domain*

Consider the translation of body domain by the infinitesimal vector $\delta\boldsymbol{\varphi} = \delta\mathbf{a}$, so that

$$\mathbf{x}^* = \mathbf{x}+\delta\mathbf{a}. \tag{17}$$

The surface tractions $\mathbf{T}^0$ and boundary displacements $\mathbf{u}^0$ are also translated correspondingly, thus

$$\begin{aligned}\delta\mathbf{T}^0 = \delta\sigma\cdot\mathbf{n}+\sigma\cdot\delta\mathbf{n} = (\delta\bar{\sigma}+\sigma_{,k}\delta a_k)\cdot\mathbf{n} = 0 \quad \text{on} \quad S_T,\\ \delta\mathbf{u}^0 = \delta\bar{\mathbf{u}}^0+\mathbf{u}_{,k}\delta a_k = 0 \quad \text{on} \quad S_u.\end{aligned} \tag{18}$$

The local variations $\delta\bar{T}^0_i$ and $\delta\bar{u}^0_i$ are therefore expressed as follows:

$$\delta\bar{T}^0_i = -\sigma_{ij,k}n_j\delta a_k \quad \text{on} \quad S_T, \qquad \delta\bar{u}^0_i = -u_{i,k}\delta a_k \quad \text{on} \quad S_u. \tag{19}$$

The expression (14) for δG_1 now takes the form

$$\begin{aligned}\delta G_1 = \Big\{\int\Big[\psi n_k\frac{\partial h}{\partial u_i}u_{i,k} + \frac{\partial h}{\partial T_i}\sigma_{ij,k}n_j\Big]\mathrm{d}S - \int\Big(\frac{\partial h}{\partial u_i}-\sigma^r_{ij}n_j\Big)u_{i,k}\,\mathrm{d}S_u\\ -\int\Big(\frac{\partial h}{\partial T_i}+u^a_i\Big)\sigma_{ij,k}n_j\,\mathrm{d}S_T\Big\}\delta a_k = \Big\{\int[\psi\delta_{kj}+\sigma^r_{ij}u_{i,k}-\sigma_{ij,k}u^a_i]n_j\,\mathrm{d}S\Big\}\delta a_k\\ +\Big\{\int\Big(\frac{\partial h}{\partial u_i}-\sigma^r_{ij}n_j\Big)u_{i,k}\,\mathrm{d}S_T + \int\Big(\frac{\partial h}{\partial T_i}+u^a_i\Big)\sigma_{ij,k}n_j\,\mathrm{d}S_u\Big\}\delta a_k.\end{aligned} \tag{20}$$

The last two integrals of (20) vanish in view of (9), and finally the variation δG_1 is expressed in the form

$$\delta G_1 = (Z^k_{1T})_S\delta a_k, \qquad (k = 1,2,3), \tag{21}$$

where

$$(Z^k_{1T})_S = \int [\psi\delta_{kj} + \sigma^r_{ij}u_{i,k} - \sigma_{ij,k}u^a_i]n_j \, dS. \tag{22}$$

Using the equality

$$\int \sigma_{ij}u^a_{i,j}n_k \, dS = \int \sigma_{ij,k}u^a_i n_j \, dS + \int \sigma_{ij}u^a_{i,j}n_j \, dS, \tag{23}$$

the expression for Z^k_{1T} can be presented in the equivalent form

$$(Z^k_{1T})_S = \int [\psi\delta_{kj} - \sigma_{il}\varepsilon^a_{il}\delta_{kj} + \sigma_{ij}u^a_{i,k} + \sigma^r_{ij}u_{i,k}]n_j \, dS, \tag{24}$$

containing only gradients of the displacement fields u_i and u^a_i.

THEOREM 1: For a linear elastic and *homogeneous* body, the integral $(Z^k_{1T})_S$ vanishes for any closed surface within the body, thus

$$(Z^k_{1T})_S = \int [\psi\delta_{kj} - \sigma_{ij,k}u^a_i + \sigma^r_{ij}u_{i,k}]n_j \, dS = 0, \qquad (k = 1, 2, 3). \tag{25}$$

To prove this theorem, let us transform (22) into a volume integral and use (9)–(12), obtaining

$$\begin{aligned}
\int [\psi\delta_{kj} - \sigma_{ij,k}u^a_i + \sigma^r_{ij}u_{i,k}]n_j \, dS &= \int [\psi_{,k} - (\sigma_{ij,k}u^a_i)_{,j} \\
+ (\sigma^r_{ij}u_{i,k})_{,j}] \, dV &= \int \left[\frac{\partial\psi}{\partial\sigma_{ij}}\sigma_{ij,k} + \frac{\partial\psi}{\partial u_i}u_{i,k} - \sigma_{ij,jk}u^a_i - \sigma_{ij,k}u^a_{i,j}\right. \\
+ \sigma^r_{ij,j}u_{i,k} + \sigma^r_{ij}u_{i,jk}] \, dV &= \int [\varepsilon^i_{ij}\sigma_{ij,k} + f^a_i u_{i,k} - \sigma_{ij,k}\varepsilon^a_{ij} \\
- f^a_i u_{i,k} + \sigma^r_{ij}\varepsilon_{ij,k}] \, dV &= \int [-\varepsilon^r_{ij}\sigma_{ij,k} + \sigma^r_{ij}\varepsilon_{ij,k}] \, dV = 0,
\end{aligned} \tag{26}$$

since for a homogeneous body

$$\begin{aligned}
\sigma^r_{ij}\varepsilon_{ij,k} = \sigma^r_{ij}(C_{ijpq}\sigma_{pq})_{,k} &= \sigma^r_{ij}C_{ijpq}\sigma_{pq,k} = \varepsilon^r_{pq}\sigma_{pq,k}, \\
(C_{ijpq})_{,k} &= (D_{ijpq})_{,k} = 0.
\end{aligned} \tag{27}$$

Here C denotes the elastic compliance matrix of both the primary and adjoint bodies.

For a non-homogeneous body, the integral (22) according to (21) represents the variation of the functional G_1 due to infinitesimal translation of the boundary with respect to inhomogeneity. Alternatively, we can consider the translation of inhomogeneity or internal void with the exterior boundary fixed, cf. Fig. 2. In Fig. 2(a), the exterior boundary does not vary and the void of surface S_0 translates through the distance $\delta\mathbf{a}$ within the homogeneous material. The variation of G_1 can now be calculated from (21) by considering the integral (22) or (24) along the void surface S_0. For the free surface S_0 the expression (24) is simplified, namely

$$(Z^k_{1T})_{S_0} = \int (\psi - \sigma_{ij}\varepsilon^a_{ij})n_k \, dS_0. \tag{28}$$

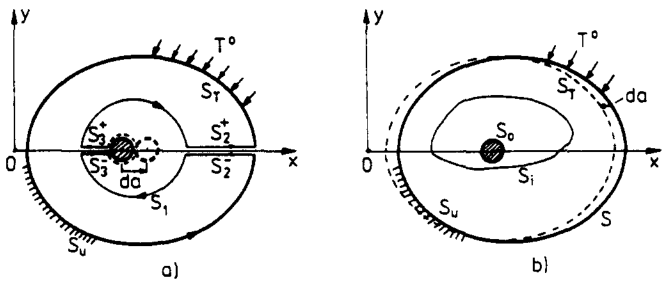

Fig. 2. (a) Translation of inhomogeneity within the body. (b) Translation of body with respect to fixed inhomogeneity.

Consider now the arbitrary closed surface S_1 enclosing the cavity and connect it to the cavity surface S_0 by the cuts S_3^+ and S_3^-. Since the integral Z_{1T}^k taken along the surface $S_1+S_3^++S_0+S_3^-$ vanishes and the integrals along S_3^+ and S_3^- cancel, we obtain

$$(Z_{1T}^k)_{S_0} = (Z_{1T}^k)_{S_1} = (Z_{1T}^k)_S. \tag{29}$$

The transition from S_1 to S can be performed by cuts S_2^+ and S_2^-.

An alternative way to calculate the variation of G_1 is to consider the translation of the body domain through the vector $\delta\mathbf{a}$ with the cavity fixed in space, cf. Fig. 2(b). The transition from the boundary surface S to an arbitrary closed surface S_1 enclosing the cavity or to the cavity surface S_0 is obtained by considering the cuts between these surfaces.

The variation of the functional G_2 is expressed similarly as in the previous case. Introduce the adjoint body of the same shape and elastic stiffness or compliance matrices, but satisfying the boundary conditions (9) and with imposed body force and initial stress fields specified by (15). Starting from (16), we obtain

$$\delta G_2 = (Z_{2T}^k)_S \delta a_k, \tag{30}$$

where

$$(Z_{2T}^k)_S = \int [\phi\delta_{kj} - \sigma_{ij,k}u_i^a + \sigma_{ij}^r u_{i,k}]n_j \, \mathrm{d}S, \qquad (k = 1,2,3). \tag{31}$$

Thus Z_{2T}^k is expressed analogously as Z_{1T}^k with $\psi(\boldsymbol{\sigma}, \mathbf{u})$ replaced by $\phi(\boldsymbol{\varepsilon}, \mathbf{u})$. Theorem 1 applies for the integral (31); thus

$$(Z_{2T}^k)_S = 0 \tag{32}$$

for any closed surface within the homogeneous body.

3.2. *Rotation of body domain*

Consider now the case where the body is rotated in the vicinity of its equilibrium position, and denote the infinitesimal rotation vector by $\delta\omega_p$. The external tractions and surface displacements are also rotated correspondingly. The variation of point position is given by

$$x_k^* = x_k + \delta\varphi_k = x_k + e_{kpl}x_l\delta\omega_p = (\delta_{kl} + e_{kpl}\delta\omega_p)x_l = \alpha_{kl}x_l, \tag{33}$$

and

$$\delta\varphi_k = e_{kpl}x_l\delta\omega_p, \qquad \alpha_{kl} = \delta_{kl} + e_{kpl}\delta\omega_p, \tag{34}$$

where e_{kpl} denotes the permutation tensor. The variation of the unit vector $\mathbf{n}$ normal to the boundary surface S is expressed as follows, cf. Ref. [16]:

$$\delta n_j = n_j n_k n_l \delta\varphi_{k,l} - n_k \delta\varphi_{k,j} = (e_{kpl} n_j n_k n_l - e_{kpj} n_k)\delta\omega_p = -e_{kpj} n_k \delta\omega_p. \tag{35}$$

The variation of the displacement field is

$$\delta u_i = \delta\bar{u}_i + u_{i,k}\delta\varphi_k = \delta\bar{u}_i + e_{kpl} u_{i,k} x_l \delta\omega_p. \tag{36}$$

On the other hand, when the displacement field is rotated, we have

$$\delta u_i = e_{ipl} u_l \delta\omega_p. \tag{37}$$

From (36) and (37), it follows that

$$\delta\bar{u}_i^0 = (e_{ipl} u_l - e_{kpl} u_{i,k} x_l)\delta\omega_p \qquad \text{on} \quad S_u. \tag{38}$$

On the loaded boundary, we have

$$\delta T_i^0 = e_{ipl} T_l^0 \delta\omega_p = e_{ipl}\sigma_{lj} n_j \delta\omega_p, \tag{39}$$

and since

$$\delta T_i^0 = \delta(\sigma_{ij} n_j) = \delta\sigma_{ij} n_j + \sigma_{ij}\delta n_j = \delta\bar{\sigma}_{ij} n_j + \sigma_{ij,k} n_j \delta\varphi_k + \sigma_{ij}\delta n_j, \tag{40}$$

we obtain on S_T

$$\begin{aligned} \delta\bar{T}_i^0 = \delta\bar{\sigma}_{ij} n_j &= \delta T_i^0 - \sigma_{ij,k} n_j \delta\varphi_k - \sigma_{ij}\delta n_j = e_{ipl}\sigma_{lj} n_j \delta\omega_p \\ &- e_{kpl}\sigma_{ij,k} x_l n_j \delta\omega_p + e_{kpj}\sigma_{ij} n_k \delta\omega_p = [e_{ipl}\sigma_{lj} n_j \\ &+ e_{kpl}(\sigma_{il} n_k - x_l \sigma_{ij,k} n_j)]\delta\omega_p. \end{aligned} \tag{41}$$

Using (33)–(41), the expression (14) for δG_1 can be presented in the form

$$\begin{aligned} \delta G_1 = e_{kpl}\Bigg\{ &\int\left[\psi x_l n_k + \frac{\partial h}{\partial u_i} u_{i,k} x_l + \frac{\partial h}{\partial T_i}\sigma_{ij,k} x_l n_j - \frac{\partial h}{\partial T_i}\sigma_{il} n_k\right] \mathrm{d}S \\ &+ \int\left[\left(\frac{\partial h}{\partial u_k} - \sigma^r_{kj} n_j\right)u_l - \left(\frac{\partial h}{\partial u_i} - \sigma^r_{ij} n_j\right)u_{i,k} x_l\right] \mathrm{d}S_u + \int\left[\left(\frac{\partial h}{\partial T_k} + u^a_k\right)\sigma_{lj} n_j \right. \\ &\left. + \left(\frac{\partial h}{\partial T_i} + u^a_i\right)(\sigma_{il} n_k - x_l \sigma_{ij,k} n_j)\right] \mathrm{d}S_T \Bigg\}\delta\omega_p \\ = \Bigg\{ e_{kpl} &\int [\psi x_l \delta_{jk} + \sigma_{lj} u^a_k + \delta_{jk}\sigma_{il} u^a_i - x_l \sigma_{ij,k} u^a_i - \sigma^r_{kj} u_l \\ &+ x_l \sigma^r_{ij} u_{i,k}] n_j \, \mathrm{d}S + e_{kpl}\int\left[\left(\frac{\partial h}{\partial u_i} - T^{a0}_i\right)u_{i,k} x_l + T^{a0}_k u_l + \frac{\partial h}{\partial T_k} T^0_l\right] \mathrm{d}S_T \\ &+ e_{kpl}\int\left[\left(\frac{\partial h}{\partial T_i} + u^{a0}_i\right)\sigma_{ij,k} x_l n_j - \left(\frac{\partial h}{\partial T_i} + u^{a0}_i\right)\sigma_{il} n_k + \frac{\partial h}{\partial u_k} u^0_l \right. \\ &\left. - T_l u^{a0}_k\right] \mathrm{d}S_u \Bigg\}\delta\omega_p. \end{aligned} \tag{42}$$

Assume now that $h = h(u_k, T_k)$ is an isotropic function of its arguments, thus

$$h = h(M_1, M_2, M_3), \tag{43}$$

where

$$M_1 = T_k T_k, \qquad M_2 = u_k u_k, \qquad M_3 = T_k u_k \tag{44}$$

are invariants of $\mathbf{u}$ and $\mathbf{T}$. In view of (43), we have

$$\frac{\partial h}{\partial T_k} = \frac{\partial h}{\partial M_1} 2T_k + \frac{\partial h}{\partial M_3} u_k, \qquad \frac{\partial h}{\partial u_k} = \frac{\partial h}{\partial M_3} T_k + \frac{\partial h}{\partial M_2} 2u_k. \tag{45}$$

Substituting (45) into (42) and accounting for boundary conditions (9), it is seen that the last two integrals of (42) vanish and the variation δG_1 can be presented in the form

$$\delta G_1 = (Z^p_{1R})_S \delta\omega_p, \tag{46}$$

where

$$(Z^p_{1R})_S = e_{kpl} \int [\psi x_l \delta_{jk} + \sigma_{ij} u^a_k + \delta_{jk} \sigma_{il} u^a_i - x_l \sigma_{ij,k} u^a_i - \sigma^r_{kj} u_l + x_l \sigma^r_{ij} u_{i,k}] n_j \, \mathrm{d}S. \tag{47}$$

Using the equalities

$$e_{kpl} \int \sigma_{ij,k} u^a_i x_l n_j \, \mathrm{d}S = e_{kpl} \int (\sigma_{ij} u^a_{i,j} x_l n_k - \sigma_{ij} u^a_{i,k} x_l n_j + \sigma_{il} u^a_i n_k) \, \mathrm{d}S \tag{48}$$

and

$$e_{kpl} \sigma^r_{kj} u_l n_j = -e_{kpl} \sigma^r_{lj} u_k n_j \tag{49}$$

the expression (47) for Z^p_{1R} can be presented in an equivalent form containing only gradients of displacement fields u_i and u^a_i, namely

$$(Z^p_{1R})_S = e_{kpl} \int [-\sigma_{iq} \varepsilon^a_{iq} x_l \delta_{kj} + \sigma_{ij} u^a_{i,k} x_l + \sigma_{lj} u^a_k + \sigma^r_{lj} u_k + \sigma^r_{ij} u_{i,k} x_l + \psi x_l \delta_{kj}] n_j \, \mathrm{d}S. \tag{50}$$

The expression (50) for Z^p_{1R} can further be transformed into the volume integral

$$\begin{aligned}
(Z^p_{1R})_S = e_{kpl} \int [&(\sigma_{lj} u^a_k)_{,j} + (\sigma_{il} u^a_i)_{,k} - (x_l \sigma_{ij,k} u^a_i)_{,j} - (\sigma^r_{kj} u_l)_{,j} \\
&+ (x_l \sigma^r_{ij} u_{i,k})_{,j} + (\psi x_l)_{,k}] \, \mathrm{d}V = e_{kpl} \int [\sigma_{lj} u^a_{k,j} + \sigma_{il,k} u^a_i + \sigma_{il} u^a_{i,k} \\
&- \sigma_{il,k} u^a_i - x_l \sigma_{ij,k} u^a_{i,j} - \sigma^r_{kj,j} u_l - \sigma^r_{kj} u_{l,j} + \sigma^r_{il} u_{i,k} \\
&+ x_l \sigma^r_{ij,j} u_{i,k} + x_l \sigma^r_{ij} u_{i,kj} + \psi \delta_{lk} + \varepsilon^a_{ij} \sigma_{ij,k} x_l + f^a_i u_{i,k} x_l] \, \mathrm{d}V \\
= e_{kpl} \int &[2\sigma_{il} \varepsilon^a_{ik} + f^a_k u_l - \sigma^r_{ki} u_{l,i} - \sigma^r_{ik} u_{i,k}] \, \mathrm{d}V \\
= e_{kpl} \int &[2\sigma_{il} \varepsilon^a_{ik} + f^a_k u_l - 2\sigma^r_{ik} \varepsilon_{il}] \, \mathrm{d}V.
\end{aligned} \tag{51}$$

In writing (50), it is assumed that the body considered is *homogeneous*. Using now Hooke's law for an isotropic body,

$$\sigma^r_{ik} = \lambda \varepsilon^r_{mm} \delta_{ik} + 2\mu \varepsilon^r_{ik}, \tag{52}$$

where λ and μ are Lame constants, we obtain

$$e_{kpl}\sigma^r_{ik}\varepsilon_{il} = e_{kpl}(\lambda\varepsilon^r_{mm}\delta_{kl}+2\mu\varepsilon^r_{ik})\varepsilon_{ii} = e_{kpl}2\mu\varepsilon^r_{ik}\varepsilon_{il}$$

$$= e_{kpl}2\mu\varepsilon^r_{ik}\left(-\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{mm}\delta_{il}+\frac{1}{\mu}\sigma_{il}\right) = e_{kpl}\varepsilon^r_{ik}\sigma_{il}, \quad (53)$$

and the expression for Z^p_{1R} takes the form

$$(Z^p_{1R})_S = e_{kpl}\int(2\sigma_{il}\varepsilon^a_{ik}+f^a_k u_l-2\sigma_{il}\varepsilon^r_{ik})\,\mathrm{d}V = e_{kpl}\int(2\varepsilon^i_{ik}\sigma_{il}+f^a_k u_l)\,\mathrm{d}V. \quad (54)$$

Assume now that $\psi(\boldsymbol{\sigma},\mathbf{u}) = \psi_1(\boldsymbol{\sigma})+\psi_2(\mathbf{u})$ where $\psi_1(\boldsymbol{\sigma})$ and $\psi_2(\mathbf{u})$ are *isotropic* functions of stress and displacement; thus,

$$\psi_1 = \psi_1(\sigma_{ij}) = \psi_1(J_1,J_2,J_3), \qquad \psi_2 = \psi_2(u_i) = \psi_2(I_u), \quad (55)$$

where J_1, J_2, J_3 are the stress tensor invariants and I_u is the displacement vector invariant, that is

$$J_1 = \sigma_{ii}, \qquad J_2 = \tfrac{1}{2}(\sigma_{ij}\sigma_{ji}-\sigma_{ii}\sigma_{jj}),$$
$$J_3 = \tfrac{1}{6}e_{mnq}e_{rst}\sigma_{mr}\sigma_{ns}\sigma_{qt}, \quad (56)$$
$$I_u = u_iu_i.$$

With these assumptions, the following equalities now occur:

$$\varepsilon^i_{ik} = \frac{\partial\psi_1}{\partial\sigma_{ik}} = \frac{\partial\psi_1}{\partial J_m}\frac{\partial J_m}{\partial\sigma_{ik}}, \qquad f^a_k = \frac{\partial\psi_2}{\partial u_k} = \frac{\partial\psi_2}{\partial I_u}\frac{\partial I_u}{\partial u_k} = 2\frac{\partial\psi_2}{\partial I_u}u_k, \quad (57)$$

where

$$\frac{\partial J_1}{\partial\sigma_{ik}} = \delta_{ik}, \quad \frac{\partial J_2}{\partial\sigma_{ik}} = \sigma_{ik}-\delta_{ik}\sigma_{qq}, \quad \frac{\partial J_3}{\partial\sigma_{ik}} = \tfrac{1}{2}e_{iqr}e_{kst}\sigma_{qs}\sigma_{rt}. \quad (58)$$

The expression for Z^p_{1R} can be presented in the form

$$(Z^p_{1R})_S = 2e_{kpl}\int\left\{\left[\frac{\partial\psi}{\partial J_1}\delta_{ik}+\frac{\partial\psi}{\partial J_2}(\sigma_{ik}-\delta_{ik}\sigma_{qq})+\frac{\partial\psi}{\partial J_3}\frac{1}{2}e_{iqr}e_{kst}\sigma_{qs}\sigma_{rt}\right]\sigma_{il}\right.$$
$$\left.+\frac{\partial\psi}{\partial I_u}u_ku_l\right\}\mathrm{d}V = 2\int\left\{e_{kpl}\left[\frac{\partial\psi}{\partial J_1}\sigma_{kl}+\frac{\partial\psi}{\partial J_2}(\sigma_{ik}\sigma_{il}-\sigma_{kl}\sigma_{qq})+\frac{\partial\psi}{\partial I_u}u_ku_l\right]\right.$$
$$\left.+\frac{\partial\psi}{\partial J_3}e_{iqr}\sigma_{pq}\sigma_{rl}\sigma_{il}\right\}\mathrm{d}V = 0. \quad (59)$$

Thus, the following theorem can now be stated.

THEOREM 2: For a linear elastic and isotropic body, the surface integral

$$(Z^p_{1R})_S = e_{kpl}\int[\sigma_{lj}u^a_k+\delta_{jk}\sigma_{il}u^a_i-x_l\sigma_{ij,k}u^a_i-\sigma^r_{kj}u_l+x_l\sigma^r_{ij}u_{i,k}+\delta_{jk}\psi x_1]n_j\,\mathrm{d}S \quad (60)$$

vanishes for any closed surface within the body, provided the adjoint system satisfies (9) and (10) and the functions $\psi = \psi_1(\boldsymbol{\sigma})+\psi_2(\mathbf{u})$ and $h(\mathbf{u},\mathbf{T})$ are isotropic functions of their arguments.

The variation of the functional G_2 is expressed similarly to (46) and (47) with $\psi(\boldsymbol{\sigma}, \mathbf{u})$ replaced by $\phi(\boldsymbol{\varepsilon}, \mathbf{u}) = \phi_1(\boldsymbol{\varepsilon}) + \phi_2(\mathbf{u})$, thus

$$\delta G_2 = (Z^p_{2R})_S \delta\omega_p, \tag{61}$$

where Z^p_{2R} is expressed identically as (50) with $\psi_1(\boldsymbol{\sigma})$ replaced by $\phi_1(\boldsymbol{\varepsilon})$.

3.3. *Expansion or contraction of body domain*

In this section, we shall confine ourselves to particular cases of general functionals (1) and (2). Consider first the transformation

$$x_k^* = (1+\delta\eta)x_k = (1+\delta\eta)\delta_{ki}x_i, \qquad \delta\varphi_k = x_k\delta\eta. \tag{62}$$

The matrix of coordinate transformation is now

$$\alpha_{ki} = (1+\delta\eta)\delta_{ki}. \tag{63}$$

The variation of the exterior normal to the boundary surface now vanishes, thus

$$\delta n_j = n_j n_k n_l \delta\varphi_{k,l} - n_k \delta\varphi_{k,j} = (n_j n_k n_l \delta_{kl} - n_k \delta_{kj})\delta\eta = n_j(n_k n_k - 1)\delta\eta = 0, \tag{64}$$

and the variation of the displacement field is

$$u_i^* = (1+\xi\delta\eta)u_i, \qquad \delta u_i = \xi u_i \delta\eta = \delta\bar{u}_i + u_{i,k}x_k\delta\eta, \tag{65}$$

so that

$$\delta\bar{u}_i = (\xi u_i - x_k u_{i,k})\delta\eta, \tag{66}$$

where ξ is a constant to be determined. The variation of strain field follows from (66), namely

$$\begin{gathered}\frac{\partial u_i^*}{\partial x_j^*} = \frac{\partial[(1+\xi\delta\eta)u_i]}{\partial[(1+\delta\eta)x_j]} = \frac{1+\xi\delta\eta}{1+\delta\eta}\frac{\partial u_i}{\partial x_j} \cong [1+(\xi-1)\delta\eta]\frac{\partial u_i}{\partial x_j},\\ \varepsilon_{ij}^* = [1+(\xi-1)\delta\eta]\varepsilon_{ij}, \quad \delta\varepsilon_{ij} = (\xi-1)\varepsilon_{ij}\delta\eta = \delta\bar{\varepsilon}_{ij} + x_k\varepsilon_{ij,k}\delta\eta,\end{gathered} \tag{67}$$

and

$$\delta\bar{\varepsilon}_{ij} = [(\xi-1)\varepsilon_{ij} - x_k\varepsilon_{ij,k}]\delta\eta. \tag{68}$$

The stress variation is expressed similarly, thus

$$\begin{gathered}\sigma_{ij}^* = [1+(\xi-1)\delta\eta]\sigma_{ij}, \qquad \delta\bar{\sigma}_{ij} = [(\xi-1)\sigma_{ij} - x_k\sigma_{ij,k}]\delta\eta,\\ \delta\bar{\sigma}_{ij}n_j = [(\xi-1)\sigma_{ij} - x_k\sigma_{ij,k}]n_j\delta\eta.\end{gathered} \tag{69}$$

Consider now the particular form of the functional (1) for which $\psi(\boldsymbol{\sigma}, \mathbf{u}) = \psi_1(\boldsymbol{\sigma})$, $h(\mathbf{u}, \mathbf{T}) = 0$, thus

$$G_\sigma = \int \psi_1(\boldsymbol{\sigma})\, \mathrm{d}V. \tag{70}$$

Assume $\psi(\boldsymbol{\sigma})$ to be a homogeneous function of stress of order p; thus,

$$\psi_1(t\boldsymbol{\sigma}) = t^p\psi_1(\boldsymbol{\sigma}), \tag{71}$$

and

$$\psi_1(\sigma^*) = [1+(\xi-1)\delta\eta]^p\psi_1(\sigma),$$
$$\delta\psi_1(\sigma) = \{[1+(\xi-1)\delta\eta]^p-1\}\psi_1(\sigma). \tag{72}$$

The constant ξ is now determined by requiring the invariance of $\psi\, dV$ under the transformation (62), that is

$$\delta(\psi_1\, dV) = \delta\psi_1\, dV + \psi_1\delta(dV) = 0, \tag{73}$$

where $\delta(dV) = 3\delta\eta\, dV$ for a three-dimensional case and $\delta(dV) = 2\delta\eta\, dV$ for the plane case. In view of (71), the invariance condition for the three-dimensional scale change is expressed as follows:

$$\{[1+(\xi-1)\delta\eta]^p-1\}\psi_1\, dV + 3\delta\eta\psi_1\, dV = 0. \tag{74}$$

Considering only linear terms of $\delta\eta$, from (74) we obtain

$$1+p(\xi-1)\delta\eta-1+3\delta\eta = 0, \qquad \xi = \frac{p-3}{p}. \tag{75}$$

Similarly, for the plane case, we have

$$[1+p(\xi-1)\delta\eta]^p-1+2\delta\eta = 0, \qquad \xi = \frac{p-2}{p}. \tag{76}$$

The variation of G_σ is now expressed as follows in the case of three-dimensional problem:

$$\delta G_\sigma = (Z_{\sigma E})_S\delta\eta, \tag{77}$$

where

$$(Z_{\sigma E})_S = \iint\left[-\frac{3}{p}\sigma_{ij}u_i^a - x_k\sigma_{ij,k}u_i^a - \frac{p-3}{p}\sigma_{ij}^r u_i + x_k\sigma_{ij}^r u_{i,k} + \delta_{jk}x_k\psi_1\right]n_j\, dS. \tag{78}$$

The eqivalent form of $Z_{\sigma E}$ containing only gradients of displacement fields u_i and u_i^a can be expressed as follows:

$$(Z_{\sigma E})_S = \iint\left[\frac{2p-3}{p}\sigma_{ij}u_i^a - x_k\sigma_{il}\varepsilon_{il}^a\delta_{kj} + x_k\sigma_{ij}u_{i,k}^a - \frac{p-3}{p}\sigma_{ij}^r u_i + x_k\sigma_{ij}^r u_{i,k} + \delta_{jk}x_k\psi_1\right]n_j\, dS. \tag{79}$$

Now, let us demonstrate that the integral (78) vanishes for any closed surface S within a homogeneous body. Transforming (78) into a volume integral, we obtain

$$\begin{aligned}(Z_{\sigma E})_S &= \iint\left[-\frac{3}{p}(\sigma_{ij}u_i^a)_{,j} - (x_k\sigma_{ij,k}u_i^a)_{,j} - \frac{p-3}{p}(\sigma_{ij}^r u_i)_{,j} + (x_k\sigma_{ij}^r u_{i,k})_{,j} + (x_k\psi_1)_{,k}\right] dV\\ &= \iint\left[-\frac{3}{p}\sigma_{ij}\varepsilon_{ij}^a - x_k\sigma_{ij,k}\varepsilon_{ij}^a - \frac{p-3}{p}\sigma_{ij}^r\varepsilon_{ij} + \sigma_{ij}^r\varepsilon_{ij} + x_k\sigma_{ij}^r\varepsilon_{ij,k} + x_k\frac{\partial\psi_1}{\partial\sigma_{ij}}\sigma_{ij,k} + 3\psi_1\right] dV\\ &= \iint\left[-\frac{3}{p}\sigma_{ij}\varepsilon_{ij}^a - x_k(\sigma_{ij,k}\varepsilon_{ij}^a - \varepsilon_{ij}^r\sigma_{ij,k} - \varepsilon_{ij}^i\sigma_{ij,k}) + \frac{3}{p}\varepsilon_{ij}^r\sigma_{ij} + 3\psi_1\right] dV\\ &= 3\int\left(-\frac{1}{p}\varepsilon_{ij}^i\sigma_{ij} + \psi_1\right) dV = \frac{3}{p}\int\left(-\frac{\partial\psi_1}{\partial\sigma_{ij}}\sigma_{ij} + p\psi_1\right) dV = 0,\end{aligned} \tag{80}$$

since for a homogeneous function of order p we have

$$\frac{\partial \psi_1}{\partial \sigma_{ij}} \sigma_{ij} = p\psi_1(\sigma_{ij}). \tag{81}$$

Similarly, for the functional (2), we have

$$G_\varepsilon = \int \phi_1(\boldsymbol{\varepsilon})\, \mathrm{d}V, \qquad \delta G_\varepsilon = (Z_{\varepsilon E})_S \delta\eta, \tag{82}$$

where

$$(Z_{\varepsilon E})_S = \iint \left[-\frac{3}{p}\sigma_{ij}u_i^a - x_k\sigma_{ij,k}u_i^a - \frac{p-3}{p}\sigma_{ij}^r u_i + x_k\sigma_{ij}^r u_{i,k} + \delta_{jk}x_k\phi_1 \right] n_j\, \mathrm{d}S, \tag{83}$$

or $Z_{\varepsilon E}$ is expressed by (79) with $\psi(\boldsymbol{\sigma})$ replaced by $\phi(\boldsymbol{\varepsilon})$ and $(Z_{\varepsilon E})_S = 0$.

Consider now the displacement functional

$$G = \int \psi_2(\mathbf{u})\, \mathrm{d}V \tag{84}$$

specified over the body domain, where $\psi_2(\mathbf{u})$ is a homogeneous function of order p. Similarly to the previous analysis, we have

$$\psi_2(\mathbf{u}^*) = (1+\xi\delta\eta)^p\psi_2(\mathbf{u}), \qquad \delta\psi_2(\mathbf{u}) = [(1+\xi\delta\eta)^p - 1]\psi_2(\mathbf{u}). \tag{85}$$

Assuming invariance of $(\psi_2\, \mathrm{d}V)$ under the transformation (62), we obtain the values of ξ for the three-dimensional and plane cases, namely

$$(\xi)_{3\mathrm{d}} = -\frac{3}{p}, \qquad (\xi)_{\mathrm{plane}} = -\frac{2}{p}. \tag{86}$$

The local variations of boundary displacements on S_u and of surface tractions on S_T, in view of (66) and (68), are now expressed as follows for the three-dimensional transformation:

$$\begin{aligned} \delta\bar{u}_i^0 &= \left(-\frac{3}{p}u_i + x_k u_{i,k} \right)\delta\eta \quad \text{on} \quad S_u, \\ \delta\bar{T}_i^0 &= -\left(\frac{3+p}{p}\sigma_{ij} + x_k\sigma_{ij,k} \right) n_j\delta\eta \quad \text{on} \quad S_T, \end{aligned} \tag{87}$$

and the variation of G can be briefly expressed using the general formula (14), as

$$\delta G = (Z_{vE})_S \delta\eta, \tag{88}$$

where

$$(Z_{vE})_S = \iint \left[\delta_{jk}x_k\psi_2 + \frac{3}{p}\sigma_{ij}^r u_i + x_k\sigma_{ij}^r u_{i,k} - \frac{3+p}{p}\sigma_{ij}u_i^a - x_k\sigma_{ij,k}u_i^a \right] n_j\, \mathrm{d}S, \tag{89}$$

since now $\sigma_{ij}^r n_j = T_i^{a_0} = 0$ on S_T and $u_i^a = 0$ on S_u. The invariance of (89) for a homogeneous linear elastic body is shown by transforming (89) into a volume integral as follows:

$$
\begin{aligned}
(Z_{rE})_S &= \int \left[(x_k \psi_2)_{,k} + \frac{3}{p} (\sigma^r_{ij} u_i)_{,j} + (x_k \sigma^r_{ij} u_{i,k})_{,j} - \frac{3+p}{p} (\sigma_{ij} u^a_i)_{,j} - (x_k \sigma_{ij,k} u^a_i)_{,j} \right] \mathrm{d}V \\
&= \int \left[3\psi_2 + x_k f^a_i u_{i,k} - \frac{3}{p} f^a_i u_i + \frac{3}{p} \sigma^r_{ij} \varepsilon_{ij} + \sigma^r_{ij} \varepsilon_{ij} - x_k f^a_i u_{i,k} + x_k \sigma^r_{ij} \varepsilon_{ij,k} \right. \\
&\quad \left. - \frac{3+p}{p} \sigma_{ij} \varepsilon^a_{ij} - x_k \sigma_{ij,k} \varepsilon^a_{ij} \right] \mathrm{d}V = \int \left(3\psi - \frac{3}{p} f^a_i u_i \right) \mathrm{d}V = \frac{3}{p} \int \left(p\psi_2 - \frac{\partial \psi_2}{\partial u_i} u_i \right) \mathrm{d}V = 0.
\end{aligned} \tag{90}
$$

Similar expressions will be obtained by considering surface displacement and traction functionals. Consider first the functional

$$G = \int h_1(\mathbf{T}) \, \mathrm{d}S, \tag{91}$$

where $h_1(\mathbf{T})$ is assumed to be a homogeneous function of order p. We have

$$h_1(\mathbf{T}^*) = [1 + (\xi - 1)\delta\eta]^p h_1(\mathbf{T}), \quad \delta h_1(\mathbf{T}) = \{[1 + (\xi - 1)\delta\eta]^p - 1\} \, h_1(\mathbf{T}). \tag{92}$$

Assume now the invariance of $(h_1 \, \mathrm{d}S)$ under the transformation (62) and note that

$$
\begin{aligned}
\delta(h_1 \, \mathrm{d}S) &= \delta h_1 \, \mathrm{d}S + h_1 \delta(\mathrm{d}S) = 0, \\
\delta(\mathrm{d}S) &= (\delta_{kl}\delta_{kl} - n_k n_l \delta_{kl})\delta\eta = 2\delta\eta,
\end{aligned} \tag{93}
$$

and $\delta(\mathrm{d}S) = \delta\eta$ in the plane case. The first condition (93) implies that

$$(\xi)_{3\mathrm{d}} = \frac{p-2}{p}, \qquad (\xi)_{\mathrm{plane}} = \frac{p-1}{p}, \tag{94}$$

and the variation of (91) takes the form

$$\delta G = (Z_{TE})_S \delta\eta, \tag{95}$$

where

$$(Z_{TE})_S = \int \left(\frac{p-2}{p} \sigma^r_{ij} u_i + x_k \sigma^r_{ij} u_{i,k} - \frac{2}{p} \sigma_{ij} u^a_i - x_k \sigma_{ij,k} u^a_i \right) n_j \, \mathrm{d}S \tag{96}$$

for the three-dimensional case.

Similarly, for the functional

$$G = \int h_2(\mathbf{u}) \, \mathrm{d}S, \tag{97}$$

where $h_2(\mathbf{u})$ is a homogeneous function of order p, from (93) we obtain

$$(\xi)_{3\mathrm{d}} = -\frac{2}{p}, \qquad (\xi)_{\mathrm{plane}} = -\frac{1}{p}, \tag{98}$$

and

$$\delta G = (Z_{uE})_S \delta\eta, \tag{99}$$

where

$$(Z_{uE})_S = \int \left(\frac{2}{p} \sigma^r_{ij} u_i + x_k \sigma^r_{ij} u_{i,k} - \frac{p+2}{p} \sigma_{ij} u^a_i - x_k \sigma_{ij,k} u^a_i \right) n_j \, \mathrm{d}S. \tag{100}$$

It is easy to show that (96) and (100) vanish for a linear elastic and homogeneous body. In fact, transforming (100) into a volume integral, we obtain

$$\begin{aligned}(Z_{uE})_S &= \int \left(\frac{2}{p} \sigma^r_{ij} \varepsilon_{ij} + \sigma^r_{ij} \varepsilon_{ij} + x_k \sigma^r_{ij} \varepsilon_{ij,k} - \frac{p+2}{p} \sigma_{ij} \varepsilon^a_{ij} - x_k \sigma_{ij,k} \varepsilon^a_{ij} \right) \mathrm{d}V \\ &= \int \left[\frac{p+2}{p} (\sigma^r_{ij} \varepsilon_{ij} - \sigma_{ij} \varepsilon^a_{ij}) + x_k (\sigma^r_{ij} \varepsilon_{ij,k} - \sigma_{ij,k} \varepsilon^a_{ij}) \right] \mathrm{d}V = 0,\end{aligned} \tag{101}$$

and the identical expression is obtained for the surface integral (96) when $(p+2)/p$ is replaced by $2/p$.

THEOREM 3: For a linear elastic and homogeneous body, the surface integrals (79), (83), (89), (96) and (100) vanish for any closed surface within the body. The variation of the respective functionals (70), (82), (84), (91) and (97) associated with the scale change of the body therefore vanishes.

4. PATH-INDEPENDENT INTEGRALS ASSOCIATED WITH POTENTIAL AND COMPLEMENTARY ENERGY VARIATIONS

The transition to the case where (1) or (2) coincide with the complementary or potential energy of the body can be obtained by specifying the adjoint systems and using general expressions for Z_T, Z_R and Z_E. Consider first the particular case when only specified surface tractions $\mathbf{T}^0$ or only surface displacements $\mathbf{u}^0$ expend the work on the body. In the first case, $\mathbf{u}^0 = 0$ on S_u, $\mathbf{T} = \mathbf{T}^0 \neq 0$ on S_T, and the potential and complementary energies are expressed as follows:

$$\Pi_u = \int U(\boldsymbol{\varepsilon}) \, \mathrm{d}V - \int \mathbf{T}^0 \cdot \mathbf{u} \, \mathrm{d}S_T = -\int U(\boldsymbol{\varepsilon}) \, \mathrm{d}V, \tag{102}$$

and

$$\Pi_\sigma = \int W(\boldsymbol{\sigma}) \, \mathrm{d}V - \int \mathbf{T} \cdot \mathbf{u}^0 \, \mathrm{d}S_u = \int W(\boldsymbol{\sigma}) \, \mathrm{d}V, \tag{103}$$

where $U(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}$, $W(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}$ are the specific stress and strain energies per unit volume. In the second case, $\mathbf{T}^0 = 0$ on S_T and $\mathbf{u} = \mathbf{u}^0 \neq 0$ on S_u, so that

$$\Pi_u = \int U(\boldsymbol{\varepsilon}) \, \mathrm{d}V, \qquad \Pi_\sigma = -\int W(\boldsymbol{\sigma}) \, \mathrm{d}V. \tag{104}$$

Setting $h(\mathbf{T}, \mathbf{u}) = 0$, $\psi(\boldsymbol{\sigma}, \mathbf{u}) = W(\boldsymbol{\sigma})$, and $\phi(\boldsymbol{\varepsilon}, \mathbf{u}) = -U(\boldsymbol{\varepsilon})$, the functionals (1) and (2) will be equivalent to (103) and (102). Similarly, setting $\psi(\boldsymbol{\sigma}, \mathbf{u}) = -W(\boldsymbol{\sigma})$ and $\phi(\boldsymbol{\varepsilon}, \mathbf{u}) = U(\boldsymbol{\varepsilon})$ in (1) and (2), we obtain the functionals (104).

The adjoint system associated with the potential energy Π_u for the case $\mathbf{u}^0 = 0$ on S_u is now specified as follows:

$$\boldsymbol{\sigma}^i = \frac{\partial \phi}{\partial \boldsymbol{\varepsilon}} = \frac{\partial(-U)}{\partial \boldsymbol{\varepsilon}} = -\boldsymbol{\sigma} \text{ within } V, \quad \mathbf{T}^{a_0} = 0 \text{ on } S_T, \quad \mathbf{u}^{a_0} = 0 \text{ on } S_u,$$
$$\boldsymbol{\sigma}^r = 0, \quad \boldsymbol{\sigma}^a = \mathbf{D}\cdot\boldsymbol{\varepsilon}^a = -\boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}^a = -\boldsymbol{\varepsilon}, \quad \mathbf{u}^a = -\mathbf{u}, \tag{105}$$

whereas for the case $\mathbf{T}^0 = 0$ on S_T we have

$$\boldsymbol{\sigma}^i = \frac{\partial \phi}{\partial \boldsymbol{\sigma}} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma} \text{ within } V, \qquad \mathbf{T}^{a_0} = 0 \text{ on } S_T, \qquad \mathbf{u}^{a_0} = 0 \text{ on } S_u,$$
$$\boldsymbol{\sigma}^r = -\boldsymbol{\sigma}, \qquad \boldsymbol{\sigma}^a = 0, \qquad \mathbf{u}^a = \boldsymbol{\varepsilon}^a = 0. \tag{106}$$

In view of (105) and (106), the general expressions for Z^k_{2T}, Z^p_{2R} and $Z_{\varepsilon E}$ given by (31), (50) and (83) take the following forms:

$$(Z^k_{2T})_S = \int (U\delta_{kj} - \sigma_{ij}u_{i,k})n_j \,\mathrm{d}S = 0,$$
$$(Z^p_{2R})_S = e_{kpl}\int (Ux_l\delta_{kj} - \sigma_{lj}u_k - \sigma_{ij}u_{i,k}x_l)n_j \,\mathrm{d}S = 0, \tag{107}$$
$$(Z_{\varepsilon E})_S = \int (Ux_k\delta_{jk} - \sigma_{ij}u_{i,k}x_k - \tfrac{1}{2}\sigma_{ij}u_i)n_j \,\mathrm{d}S = 0.$$

The relations (107) constitute the set of conservation laws associated with invariance of the potential energy under translation, rotation and scale change of the domain of a homogeneous and isotropic body (isotropy is only required in the case of rotation). Consider now the complementary energy Π_σ. The adjoint system for the case $\mathbf{u}^0 = 0$ on S_u is now specified as follows:

$$\boldsymbol{\varepsilon}^i = \frac{\partial \psi}{\partial \boldsymbol{\sigma}} = \frac{\partial W}{\partial \boldsymbol{\sigma}} = \boldsymbol{\varepsilon} \text{ within } V, \quad \mathbf{T}^{a_0} = 0 \text{ on } S_T, \quad \mathbf{u}^{a_0} = 0 \text{ on } S_u,$$
$$\boldsymbol{\sigma}^r = 0, \qquad \boldsymbol{\sigma}^a = \mathbf{D}\cdot\boldsymbol{\varepsilon}^a = \boldsymbol{\sigma}, \qquad \boldsymbol{\varepsilon}^a = \boldsymbol{\varepsilon}, \qquad \mathbf{u}^a = \mathbf{u}, \tag{108}$$

whereas for the case $\mathbf{T}^0 = 0$ on S_T we can write

$$\boldsymbol{\varepsilon}^i = \frac{\partial \psi}{\partial \boldsymbol{\sigma}} = \frac{\partial(-W)}{\partial \boldsymbol{\sigma}} = -\boldsymbol{\varepsilon} \text{ within } V, \quad \mathbf{T}^{a_0} = 0 \text{ on } S_T, \quad \mathbf{u}^{a_0} = 0 \text{ on } S_u,$$
$$\boldsymbol{\sigma}^r = \boldsymbol{\sigma}, \qquad \boldsymbol{\sigma}^a = 0, \qquad \boldsymbol{\varepsilon}^a = \mathbf{u}^a = 0. \tag{109}$$

Using now (108) and (109) in the general expressions for Z^k_{1T}, Z^p_{1R} and $Z_{\sigma E}$ derived in the previous section, we obtain the conservation laws associated with the invariance of the complementary energy under translation, rotation and expansion of the domain of a homogeneous and isotropic body, namely

$$(Z^k_{1T})_S = \int (-W\delta_{kj} + \sigma_{ij}u_{i,k})n_j \,\mathrm{d}S = 0,$$
$$(Z^p_{1R})_S = e_{kpl}\int (-Wx_l\delta_{kj} + \sigma_{lj}u_k + \sigma_{ij}u_{i,k}x_l)n_j \,\mathrm{d}S = 0, \tag{110}$$
$$(Z_{\sigma E})_S = \int (-Wx_k\delta_{jk} + \sigma_{ij}u_{i,k}x_k + \tfrac{1}{2}\sigma_{ij}u_i)n_j \,\mathrm{d}S = 0.$$

Let us note that the conservation laws (107) and (110) are respectively equivalent to those derived earlier by Günther[3], Eshelby[1, 2], Knowles and Sternberg[4] and Bui[14].

It can be shown that the same path-independent functionals and conservation rules are obtained in the case of translation and rotation of the body when mixed boundary conditions with non-vanishing $\mathbf{u}^0$ and $\mathbf{T}^0$ on S_u and S_T are assumed.

5. CONSERVATION RULES FOR BILINEAR FUNCTIONALS

Bilinear functionals are useful in assessing the variation of local displacement or stress components due to shape variation. It was shown in Ref. [20] that they can be generated as singular cases of (1) and (2) assuming localized force or dislocation action on the adjoint body. However, it is convenient to consider this class of functionals separately. Consider a linear elastic body loaded by surface traction fields $\mathbf{T}_1^0$ and $\mathbf{T}_2^0$ on S_T, and with specified displacements $\mathbf{u}_1^0$ and $\mathbf{u}_2^0$ on S_u, respectively. Denote the corresponding state fields by $\boldsymbol{\sigma}_1$, $\boldsymbol{\varepsilon}_1$, $\mathbf{u}_1$ and $\boldsymbol{\sigma}_2$, $\boldsymbol{\varepsilon}_2$, $\mathbf{u}_2$ satisfying conditions of equilibrium, compatibility and constitutive relations. Consider now the functionals, cf. Shield and Prager[21],

$$\begin{aligned} B_1 &= \int \frac{1}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\varepsilon}_2\,\mathrm{d}V + \int \frac{1}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\varepsilon}_1\,\mathrm{d}V - \int \mathbf{T}_1^0\cdot\mathbf{u}_2\,\mathrm{d}S_T - \int \mathbf{T}_2^0\cdot\mathbf{u}_1\,\mathrm{d}S_T \\ &= \int \tilde{U}(\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)\,\mathrm{d}V - \int \mathbf{T}_1^0\cdot\mathbf{u}_2\,\mathrm{d}S_T - \int \mathbf{T}_2^0\cdot\mathbf{u}_1\,\mathrm{d}S_T, \end{aligned} \tag{111}$$

and

$$\begin{aligned} B_2 &= \int \frac{1}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\varepsilon}_2\,\mathrm{d}V + \int \frac{1}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\varepsilon}_1\,\mathrm{d}V - \int \mathbf{T}_1\cdot\mathbf{u}_2^0\,\mathrm{d}S_u - \int \mathbf{T}_2\cdot\mathbf{u}_1^0\,\mathrm{d}S_u \\ &= \int \tilde{W}(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)\,\mathrm{d}V - \int \mathbf{T}_1\cdot\mathbf{u}_2^0\,\mathrm{d}S_u - \int \mathbf{T}_2\cdot\mathbf{u}_1^0\,\mathrm{d}S_u, \end{aligned} \tag{112}$$

where $\tilde{U}(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2) = \boldsymbol{\varepsilon}_1\cdot\mathbf{D}\cdot\boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_2\cdot\mathbf{D}\cdot\boldsymbol{\varepsilon}_1$ and $\tilde{W}(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = \boldsymbol{\sigma}_1\cdot\mathbf{C}\cdot\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2\cdot\mathbf{C}\cdot\boldsymbol{\sigma}_1$ are the specific mutual strain and stress energies, so that

$$\boldsymbol{\sigma}_1 = \frac{\partial \tilde{U}}{\partial \boldsymbol{\varepsilon}_2}, \qquad \boldsymbol{\sigma}_2 = \frac{\partial \tilde{U}}{\partial \boldsymbol{\varepsilon}_1}, \qquad \boldsymbol{\varepsilon}_1 = \frac{\partial \tilde{W}}{\partial \boldsymbol{\sigma}_2}, \qquad \boldsymbol{\varepsilon}_2 = \frac{\partial \tilde{W}}{\partial \boldsymbol{\sigma}_1}. \tag{113}$$

The variation of B_1 due to shape variation is expressed similarly to (6), namely

$$\begin{aligned} \delta B_1 = &\int \left(\frac{\partial \tilde{U}}{\partial \boldsymbol{\varepsilon}_1}\cdot\delta\bar{\boldsymbol{\varepsilon}}_1 + \frac{\partial \tilde{U}}{\partial \boldsymbol{\varepsilon}_2}\cdot\delta\bar{\boldsymbol{\varepsilon}}_2\right)\mathrm{d}V + \int \tilde{U} n_k \delta\varphi_k\,\mathrm{d}S - \int \delta\mathbf{T}_1^0\cdot\mathbf{u}_2\,\mathrm{d}S_T \\ &- \int \mathbf{T}_1^0\cdot(\delta\bar{\mathbf{u}}_2 + \mathbf{u}_{2,k}\delta\varphi_k)\,\mathrm{d}S_T - \int \delta\mathbf{T}_2^0\cdot\mathbf{u}_1\,\mathrm{d}S_T \\ &- \int \mathbf{T}_2^0\cdot(\delta\bar{\mathbf{u}}_1 + \mathbf{u}_{1,k}\delta\varphi_k)\,\mathrm{d}S_T \\ &- \int (\mathbf{T}_1^0\cdot\mathbf{u}_2 + \mathbf{T}_2^0\cdot\mathbf{u}_1)(\delta_{kl} - n_k n_l)\delta\varphi_{k,l}\,\mathrm{d}S_T. \end{aligned} \tag{114}$$

Using the virtual work equation, we obtain

$$\delta B_1 = \int (\tilde{U}n_k - \mathbf{T}_1 \cdot \mathbf{u}_{2,k} - \mathbf{T}_2 \cdot \mathbf{u}_{1,k})\delta\varphi_k \, dS$$
$$- \int (\mathbf{T}_1^0 \cdot \mathbf{u}_2 + \mathbf{T}_2^0 \cdot \mathbf{u}_1)(\delta_{kl} - n_k n_l)\delta\varphi_{k,l} \, dS_T \quad (115)$$
$$- \int \delta\mathbf{T}_1^0 \cdot \mathbf{u}_2 \, dS_T - \int \delta\mathbf{T}_2^0 \cdot \mathbf{u}_1 \, dS_T + \int \mathbf{T}_1 \cdot \delta\mathbf{u}_2^0 \, dS_u + \int \mathbf{T}_2 \cdot \delta\mathbf{u}_1^0 dS_u,$$

and a similar expression is obtained for the functional B_2:

$$\delta B_2 = \int (-\tilde{W}n_k + \mathbf{T}_1 \cdot \mathbf{u}_{2,k} + \mathbf{T}_2 \cdot \mathbf{u}_{1,k})\delta\varphi_k \, dS$$
$$+ \int (\mathbf{T}_1^0 \cdot \mathbf{u}_2 + \mathbf{T}_2^0 \cdot \mathbf{u}_1)(\delta_{kl} - n_k n_l)\delta\varphi_{k,l} \, dS_T + \int \delta\mathbf{T}_1^0 \cdot \mathbf{u}_2 \, dS_T \quad (116)$$
$$+ \int \delta\mathbf{T}_2^0 \cdot \mathbf{u}_1 \, dS_T - \int \mathbf{T}_1 \cdot \delta\mathbf{u}_2^0 \, dS_u - \int \mathbf{T}_2 \cdot \delta\mathbf{u}_1^0 \, dS_u.$$

Consider now the translation of body domain: $\delta\varphi_k = \delta a_k = \text{const.}$, $\delta\mathbf{u}_1^0 = \delta\mathbf{u}_2^0 = 0$ on S_u, $\delta\mathbf{T}_1^0 = \delta\mathbf{T}_2^0 = 0$ on S_T. Similarly to (21), from (115) it follows that

$$\delta B_1 = (M_{1T}^k)_S \delta a_k, \quad (117)$$

where

$$(M_{1T}^k)_S = \int (\tilde{U}n_k - \mathbf{T}_1 \cdot \mathbf{u}_{2,k} - \mathbf{T} \cdot \mathbf{u}_{1,k}) \, dS = \int (\tilde{U}\delta_{kj} - \sigma_{1ij}u_{2i,k} - \sigma_{2ij}u_{1i,k})n_j \, dS, \quad (118)$$

and $(M_{1T}^k)_S = 0$ for any closed surface S within a homogeneous elastic body. It is seen that (118) is identical to (22) and also to the path-independent integral derived by Chen and Shield[18]. Similarly, the variation of B_2 is expressed as follows:

$$\delta B_2 = (M_{2T}^k)_S \delta a_k = -(M_{1T}^k)_S \delta a_k, \quad (119)$$

where the path-independent integral is expressed as

$$(M_{2T}^k)_S = \int (-\tilde{W}\delta_{kj} + \sigma_{1ij}u_{2i,k} + \sigma_{2ij}u_{1i,k})n_j \, dS, \quad (120)$$

and it is equivalent to (31).

The case of rotation can be treated similarly as before, and the path-independent integrals are specified by (50). It is thus seen that the variations of the bilinear functionals are expressed in terms of the same path-independent integrals as the variations of G_1 and G_2.

6. ON APPLICATION OF PATH-INDEPENDENT INTEGRALS

The derived path-independent integrals can be applied in sensitivity analysis with respect to translation, rotation or expansion of internal defects such as cracks, cavities or inclusions. In fact, the functionals (1) and (2) can be given different interpretations depending on the type of problem considered. For instance, they can represent local or averaged stress measure or distance norm between measured and theoretical values of strains and displacements within the body. Thus, these functionals can be used in identification of the positions and sizes of defects based on available measurement data. Enclosing the defect by a surface S_i, the variation of any functional G_1 or G_2 can be determined by calculating the respective path-independent integral along any surface S_i including the defect surface S_0 or the external boundary surface S. Another application, already indicated in Refs. [18, 19], would be associated with separating and determining mixed-mode singularities at crack tips.

We will now illustrate the applicability of the derived conservation rules by considering two simple examples.

Example 1. Consider a thin circular disk of external radius b and with an internal hole of radius a, loaded by the pressure p at the external perimeter $r = b$. The stress and displacement states within the disk referred to the polar coordinate system r, θ are as follows:

$$\sigma_r = k\left(\frac{a^2}{r^2} - 1\right), \quad \sigma_\theta = -k\left(\frac{a^2}{r^2} + 1\right), \quad k = \frac{pb^2}{b^2 - a^2},$$
$$u_r = -\frac{k}{E_r}\left[(1+\nu)\frac{a^2}{r^2} + (1-\nu)\right], \quad u_\theta = 0, \tag{121}$$

where E and ν denote Young's modulus and Poisson's ratio, respectively. We consider now the functional

$$G = \int \psi(\sigma)\,\mathrm{d}A = \int \sigma_e^2\,\mathrm{d}A = \int (\sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2)\,\mathrm{d}A, \tag{122}$$

and determine its variation associated with an expansion of the internal hole. According to (77) and (78), this variation can be expressed by a path-independent integral over the external perimeter. In the two-dimensional case, we have $p = 2$, $\xi = 0$, and

$$\delta G = (Z_{\sigma E})_S\delta\eta, \qquad (Z_{\sigma E})_S = b\int (\psi - \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}^a + \mathbf{T}\cdot\mathbf{u}_{,n}^a - \mathbf{T}^a\cdot\mathbf{u}_{,n})\,\mathrm{d}S_T. \tag{123}$$

The adjont body is subject to the initial strain field

$$\varepsilon_r^i = \frac{\partial\psi}{\partial\sigma_r} = 2\sigma_r - \sigma_\theta = k\left(\frac{3a^2}{r^2} - 1\right), \quad \varepsilon_\theta^i = \frac{\partial\psi}{\partial\sigma_\theta} = 2\sigma_\theta - \sigma_r = -k\left(\frac{3a^2}{r^2} + 1\right), \tag{124}$$

with the boundary conditions $T_r^a(b) = \sigma_r^r(b) = 0$, $T_r^a(a) = \sigma_r^r(a) = 0$. The solution of the adjoint problem provides the stress, strain and displacement fields

$$u_r^a = -kr\left(\frac{3a^2}{r^2} + 1\right), \qquad u_\theta^a = 0, \qquad \sigma_r^r = 0, \qquad \sigma_\theta^r = 0,$$
$$\varepsilon_r^a = k\left(\frac{3a^2}{r^2} - 1\right), \qquad \varepsilon_\theta^a = -k\left(\frac{3a^2}{r^2} + 1\right), \tag{125}$$

and

$$(Z_{\sigma E})_S = 2\pi b^2(\sigma_e^2 - \sigma_\theta\varepsilon_\theta^a) = -8\pi k^2 a^2, \qquad \delta G = -8\pi k^2 a^2\delta\eta = 8\pi k^2 a\delta a, \tag{126}$$

since $\delta a = -a\delta\eta$, where $\delta a > 0$ corresponds to the expansion of hole.

When the displacement functional

$$G = \int h(\mathbf{u})\,\mathrm{d}A = \frac{1}{2}\int u_r^2\,\mathrm{d}A \tag{127}$$

is considered, the adjoint structure is loaded by the body force field

$$f_r^a = \frac{\partial h}{\partial u_r} = u_r, \qquad f_\theta^a = 0, \tag{128}$$

and, according to (88) and (89), the variation of (127) is expressed by a surface integral over the external perimeter; thus,

$$\begin{gathered}\delta G = (Z_{vE})_S\delta\eta = -\frac{1}{a}(Z_{vE})_S\delta a,\\ (Z_{vE})_S = \int (hb - \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}^a b + \mathbf{T}\cdot\mathbf{u}^a_{,n}b - \mathbf{T}\cdot\mathbf{u}^a)\,\mathrm{d}S_T = \pi b[u_r^2(b) + 4ku_r^a(b)],\end{gathered} \tag{129}$$

where the displacement field for the adjoint body is obtained in the form

$$\begin{aligned} u_r^a &= \frac{(1-v^2)k}{2E^2}\left[\frac{1}{4}(1-v)r^3 + (1+v)a^2r\left(\ln r - \frac{1}{2}\right) + C_1 r + \frac{2}{r}C_2\right],\\ C_1 &= -\left\{\left[\frac{1}{4}\frac{(3+v)(1-v)}{1+v} + \tfrac{1}{2}(1-v)\right](a^2+b^2) + (1+v)\frac{a^2(b^2\ln b - a^2\ln a)}{b^2-a^2}\right\},\\ C_2 &= -a^2b^2\left[\frac{1}{4}(3+v) + \frac{(1+v)^2}{1-v}\frac{a^2(\ln b - \ln a)}{b^2-a^2}\right].\end{aligned} \tag{130}$$

Example 2. This example is related to generalization of the J-integral in fracture mechanics[12–14] associated with the potential energy release rate due to crack propagation. Consider a plane strain or stress case and a notch or crack with flat boundary portions oriented along the x-axis, cf. Fig. 3. For a curve Γ enclosing the notch tip and ending on flat notch surfaces, the integral (24) can be expressed as follows:

$$(Z_{1T}^1)_\Gamma = \int (\psi - \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}^a)\,\mathrm{d}y + \int\left(\mathbf{T}\cdot\frac{\partial\mathbf{u}^a}{\partial x} + \mathbf{T}^a\cdot\frac{\partial\mathbf{u}}{\partial x}\right)\mathrm{d}s, \tag{131}$$

where ds is the arc length element of Γ and dy denotes its projection on the y-axis of the x, y-Cartesian coordinate system. Similarly as for the J-integral[12], it is easy to prove path independence of this integral. In fact, considering any other curve Γ' ending at 3 and 4 on the flat notch portions, the integral over the whole closed path $\Gamma_{1-2}+2\text{–}4+\Gamma'_{4-3}+3\text{–}1$ vanishes in view of Theorem 1 since this path encloses the homogeneous material. Moreover, on flat portions 2–3 and 1–4 we have $\mathrm{d}y = 0$, $\mathbf{T} = \mathbf{T}^a = 0$ and the expression in (131) vanishes on these portions. Thus, $(Z_{1T}^1)_\Gamma = (Z_{1T}^1)_{\Gamma'}$, and the integral (131) is path-independent. A similar property occurs for the integral $(Z_{2T}^1)_\Gamma$ specified by (31). Now, if the path coincides with the curvilinear portion Γ_0 of the notch tip, the integral (131) takes the form

$$(Z_{1T}^1)_\Gamma = \int (\psi - \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}^a)\,\mathrm{d}y, \tag{132}$$

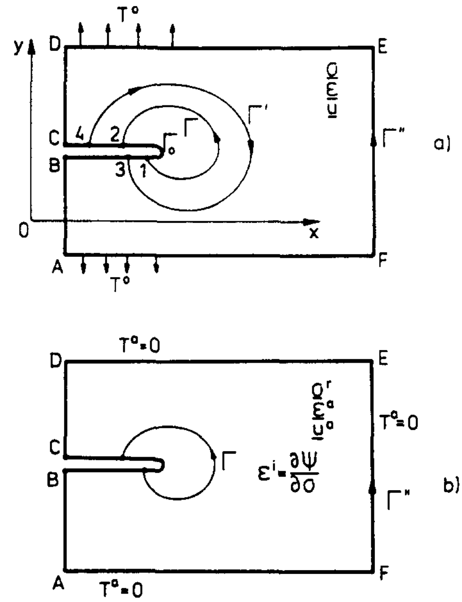

Fig. 3. Path-independent integral for a notched body in plane case: (a) body with a notch; (b) adjoint body subjected to initial strain field $\boldsymbol{\varepsilon}^i$.

and the variation of the functional (1) is given by

$$\delta G_1 = (Z\,|_T)_{\Gamma_0}\delta a = (Z\,|_T)_{\Gamma}\delta a, \tag{133}$$

where δa denotes the infinitesimal translation of the notch tip along the x-axis. The familiar property of the J-integral associated with the potential energy variation is now preserved for any stress, strain or displacement functional.

Consider, for instance, the case where the external boundary of the body coincides with the x- and y-axes, cf. Fig. 3. Assume the boundaries AD and EF to be free and DE, AF to be loaded by normal tractions. When the second term of (1) is neglected and $\psi = \psi(\boldsymbol{\sigma})$, the adjoint body is subjected to the initial strain field $\boldsymbol{\varepsilon}^i$ specified by (10) with vanishing surface tractions $\mathbf{T}^a$ on the boundary. The integral (131) now takes the form

$$(Z\,|_T)_{ADEF} = \int_{EF} [\psi(\sigma_y) - \sigma_y \varepsilon_y^a]\,\mathrm{d}y - \int_{AD} [\psi(\sigma_y) - \sigma_y \varepsilon_y^a]\,\mathrm{d}y + \int_{DE} T_y \frac{\partial u_y^a}{\partial x}\,\mathrm{d}x + \int_{AF} T_y \frac{\partial u_y^a}{\partial x}\,\mathrm{d}x. \tag{134}$$

When the body is subjected to a uniformly distributed displacement on DE and AF, then $\partial u_y^a/\partial x = 0$ and only two terms of (134) remain.

Whereas previous research effort in fracture mechanics was concentrated primarily on determining the variation of the potential energy associated with crack growth, the present method provides tools to determine variation of any functional of stress, strain or displacement. One class of functionals is associated with the problem of identification of damage or crack position and orientation. The two examples presented illustrate the idea that by calculating path-independent integrals along fixed contours, far from singularities, the variation of respective functionals can be assessed.

7. CONCLUDING REMARKS

In the present work, a new class of conservation laws and path-independent integrals was discussed. This class is associated with variation of arbitrary stress, strain and displacement functionals due to infinitesimal translation, rotation or scale change of an inhomogeneity within the elastic body. The analysis was limited to linearly elastic material within small strain theory but the extension to non-linear case is possible by following the work on sensitivity analysis in non-linear elasticity[17]. Similarly as J-integral in fracture mechanics the derived path-independent integrals Z^k_{1T} or Z^k_{2T} can be applied in studying variation of any functional due to extension of plane cracks. The bilinear functionals discussed in Section 5 provide the same path-independent integrals with proper interpretation of the adjoint system. Their application was indicated in Refs. [18, 19] to problems of determining stress intensity factors at crack tips for mixed mode conditions.

REFERENCES

1. J. D. Eshelby, The continuum theory of lattice defects. *Solid State Physics* (Edited by F. Seitz and D. Turnbull), Vol 3, p. 79. Academic Press, New York (1956).
2. J. D. Eshelby, Energy relations and energy momentum tensor in continuum mechanics. In *Inelastic Behaviour of Solids* (Edited by M. F. Kanninen *et al.*), pp. 77–115. McGraw-Hill, New York (1970).
3. W. Günther, Über einige Randintegral der Elastomechanik. *Abh. Braunschweig Wiss. Geselsch.* **14**, 54–63 (1962).
4. J. K. Knowles and E. Sternberg, On a class of conservation laws in linearized and finite elastostatics. *Arch. Rat. Mech. Anal.* **44**, 187–21 (1972).
5. D. C. Fletcher, Conservation laws in linear elastodynamics. *Arch. Rat. Mech. Anal.* **60**, 329–353 (1976).
6. E. Noether, Invariante Variationsprobleme, *Nachr. Ges. Göttingen, Math. Phys. Klasse* **2**, 235 (1918).
7. A. G. Herrmann, On conservation laws of continuum mechanics. *Int. J. Solids Struct.* **17**, 1–9 (1981).
8. G. Francfort and A. G. Herrmann, Conservation laws and material momentum in thermoelasticity. *J. Appl. Mech.* **49**, 710–714 (1982).
9. T. J. Delph, Conservation laws in linear elasticity based upon divergence transformation. *J. Elasticity* **12**, 385–393 (1982).
10. T. J. Delph, Conservation laws for materials exhibiting power-law creep. *Int. J. Solids Struct.* **19**, 907–913 (1983).
11. G. Herrmann, Some applications of invariant variational principles in mechanics of solids. In *Variational Methods in the Mechanics of Solids* (Edited by S. Nemat Nasser), pp. 145–150. Pergamon Press, Oxford (1980).
12. J. R. Rice, A path-independent integral and the approximate analysis of strain concentrations by notches and cracks. *J. Appl. Mech.* **35**, 379–386 (1968).
13. B. Budiansky and J. R. Rice, Conservation laws and energy release rates. *J. Appl. Mech.* **40**, 201–206 (1974).
14. H. D. Bui, Dual path-independent integrals in the boundary-value problems of cracks. *Engng Fract. Mech.* **6**, 287–296 (1974).
15. K. Dems and Z. Mróz, Variational approach by means of adjoint systems to structural optimization and sensitivity analysis—I. Variation of material parameters within fixed domain. *Int. J. Solids Struct.* **19**, 677–692 (1983).
16. K. Dems and Z. Mróz, Variational approach by means of adjoint systems to structural optimization and sensitivity analysis—II. Structure shape variation. *Int. J. Solids Struct.* **20**, 527–552 (1984).
17. G. Szefer, Z. Mróz and L. Demkowicz; Variational approach to sensitivity analysis in non-linear elasticity. *Arch. Mech.* Submitted for publication.
18. F. H. K. Chen and R. T. Shield, Conservation laws in elasticity of the J-integral type. *J. Appl. Math. Phys. (ZAMP)* **28**, 1–22 (1977).
19. H. D. Bui, Associated path-independent J-integrals for separating mixed mode. *J. Mech. Phys. Solids* **31**, 439–448 (1983).
20. K. Dems and Z. Mróz, Variational approach to first and second-order sensitivity analysis of elastic structures. *Int. J. Num. Meth. Engng* **21**, 637–661 (1985).
21. R. T. Shield and W. Prager, Optimal structural design for given deflection. *Z. Angew. Math. Phys.* **21**, 513–523 (1970).